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Integro-differential equation with Cauchy kernel

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Abstract

In this work, we present a Galerkin approach for solving the linear integro-differential equation of the second kind with Cauchy kernel. We use the orthogonal basis functions Legendre polynomials of the first and of the second kind. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Singular integral equations arise in many problems of mathematical physics. The mathematical formulation of physical phenomena often involves Cauchy type, or more severe, singular integral equations. Applications in many important fields, like fracture mechanics [14], elastic contact problems [3], the theory of porous filtering [13] and combined infrared radiation and molecular conduction [9], contain integral and integro-differential equation with singular kernel. The solutions of these problems may be obtained analytically using the theory developed by Mushkelishvili [16]. The books edited by Popov [19], Tricomi [18], Hochstad [12] and Green [11] contained many different methods to solve the integral and integro-differential equations analytically. In practice, we say, approximate methods are needed. So many different methods are established to obtain the solution of the integral and integro-differential equations numerically. The interested reader should consult the fine expositions in [5,8,15].

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This paper uses a classical expansion method, Legendre polynomials method, for solving the following integro-differential equation with Cauchy kernel:

$$\mu\frac{\mathrm{d}\phi}{\mathrm{d}x} - \lambda \oint_{-1}^{1} \frac{\phi(y)\,\mathrm{d}y}{x-y} = f(x), \quad |x| \leqslant 1 \tag{1.1}$$

under the condition

$$\int_{-1}^{1} \phi(y)\,\mathrm{d}y = p < \infty \quad (p \text{ is a constant}). \tag{1.2}$$

Here μ is a constant defined by the type of Eq. (1.1), if $\mu = 0$ we obtain integral equation of the first kind with Cauchy kernel, if $\mu =$ constant we have the integro-differential equation of the second kind, and if $\mu = \mu(x)$ we have the third kind. λ is a constant composed of several physical properties. Here, $\oint$ denotes integration in the principal value sense. We suppose that the unknown function $\phi(x) \in C^1([-1,1])$. Then $\oint_{-1}^{1} \phi(y)\,\mathrm{d}y/(x-y)$ exists in the principal value sense. The known function $f(x)$ is continuous in the integration interval.

The work here describes a Galerkin approach for solving the boundary value problem when $\mu = 1$. We begin the approach by transforming the initial integro-differential equation into an alternative form in which the unknown function $\phi(x)$ is receptive to approximation by an expansion technique. We used the Legendre polynomials as basis functions for the expansion. Several integral and algebraic relations and the inclusion of the orthogonality property associated with Legendre polynomials are used for determining the unknown expansion coefficients of the basis functions.

2. Information and relations

To present a regularized form of Eq. (1.1) under condition (1.2), $\mu = 1$ and following the same technique of Cauchy method [16] we obtain

$$\phi(x) = \frac{1}{\pi\sqrt{1-x^2}}\left[p - \frac{1}{\pi\lambda}\oint_{-1}^{1} \frac{\sqrt{1-\tau^2}}{\tau - x}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} - f(\tau)\right)\mathrm{d}\tau\right]. \tag{2.3}$$

The alternative expression for the constant p can be developed by multiplying (2.3) by $\sqrt{1-x^2}$ and evaluating the resultant at $x = -1$ to get

$$p = -\frac{1}{\lambda\pi}\int_{-1}^{1} \sqrt{\frac{1-\tau}{1+\tau}}\left[\frac{\mathrm{d}\phi(\tau)}{\mathrm{d}\tau} - f(\tau)\right]\mathrm{d}\tau. \tag{2.4}$$

Using (2.4) in (2.3), we have

$$\phi(x) = \frac{1}{\lambda\pi^2}\oint_{-1}^{1} \frac{\sqrt{1-\tau^2}}{\tau - x}\left[\frac{\mathrm{d}\phi}{\mathrm{d}\tau} - f(\tau)\right]\mathrm{d}\tau, \tag{2.5}$$

where we used the following integral results:

$$\int_{-1}^{1}\frac{\mathrm{d}\tau}{(\tau-x)\sqrt{1-\tau^2}}=2\int_0^{\infty}\frac{\mathrm{d}s}{1-s-(1+x)s^2}$$
$$=\frac{1}{\sqrt{1-x^2}}l_n(x)\left\{\frac{\sqrt{1-x}+s\sqrt{1+x}}{\sqrt{1-x}-s\sqrt{1+x}}\right\}\Bigg|_0^{\infty}=0,$$

where $(\tau=(1-s^2)/(1+s^2))$,

$$\int_{-1}^{1}\frac{\tau^{2n}\,\mathrm{d}\tau}{\sqrt{1-\tau^2}}=\frac{\pi(2n-1)!}{(2n)!}$$

and

$$\int_{-1}^{1}\frac{\tau^{m}\,\mathrm{d}\tau}{\sqrt{1-\tau^2}(\tau-x)}=\begin{cases}\pi l_n(x), & m=2n,\\ \pi x l_n(x), & m=2n+1,\end{cases}$$

where

$$l_n(x)=\sum_{h=0}^{n}\frac{(2h-1)!}{(2h)!}x^{2n-2h}.$$

If we substitute the value of p from (1.2) and use (2.5), we can rewrite (2.4) in the following form:

$$\int_{-1}^{1}\phi(y)\,\mathrm{d}y=-\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-\tau}{1+\tau}}\oint_{-1}^{1}\frac{\phi(z)}{\tau-z}\,\mathrm{d}z\,\mathrm{d}\tau.$$

So, we have

$$\int_{-1}^{1}\sqrt{\frac{1-\tau}{1+\tau}}\frac{\mathrm{d}\tau}{\tau-z}=-\pi.$$

Some alternative formulations can be stated, the Fredholm integral equation of the second kind with logarithmic kernel can be obtained by integrating the integral term of Eq. (1.1) by parts, to obtain

$$\mu\phi'(x)-\lambda\int_{-1}^{1}\phi'(\xi)\ln\left(\frac{|\xi-x|}{2}\right)\mathrm{d}\xi=g(x),\quad (|x|\leqslant 1),$$

where

$$g(x)=f(x)+\phi(-1)\ln\left(\frac{1+x}{2}\right)+\phi(1)\ln\left(\frac{1-x}{2}\right). \tag{2.6}$$

To guarantee the existence of a unique solution of (2.6) with condition (1.2) the discontinuous kernel must satisfy the relation

$$|\mu|>|\lambda|\left(\int_{-1}^{1}\int_{-1}^{1}\ln^2\left(\frac{|\tau-x|}{2}\right)\mathrm{d}z\,\mathrm{d}\tau\right)^{1/2}.$$

Using the famous relation [17]

$$\ln(|x-y|)=h(x,y)|x-y|^{-\alpha},\quad 0<\alpha<1, \tag{2.7}$$

where $h(x,y)\in C[-1,1]$ for all $x\in[-1,1]$, Eq. (2.6) with logarithmic kernel and with Karlman kernel (2.7) is solved numerically in [2]. The importance of Karlman kernel comes from the work of Arutiunian [4] who shows that the problem of the nonlinear theory of elasticity in its first approximation, can be reduced to Fredholm integral equation of the first kind with Karlman kernel.

3. Method of the solution

Assume the solution of Eq. (1.1) subject to condition (1.2) in the series polynomials

$$\phi(x) = \sum_{n=0}^{\infty} C_n P_n(x), \tag{3.8}$$

where C_n, the unknown expansion coefficients are to be determined, say, either by a Collocation or Galerkin method, and $P_n(x)$ are the Legendre polynomials which satisfy the orthogonal relation (see [6])

$$\int_{-1}^{1} P_n(x)P_m(x)\,\mathrm{d}x = \begin{cases} 0, & n \neq m, \\ \dfrac{2}{2n+1}, & n = m. \end{cases}$$

This series is uniformly convergence if $\phi(x) \in L_2([-1,1])$.
Differentiating (3.8), we have

$$\phi'(x) = -\sum_{n=1}^{\infty} C_n(1-x^2)^{-1/2}P_n^1(x). \tag{3.9}$$

So we write the free term of (1.1) in the form

$$f(x) = -\sum_{n=1}^{\infty} f_n(1-x^2)^{-1/2}P_n^1(x) \tag{3.10}$$

and using the famous relation (see [6,7, pp. 835])

$$Q_n(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(y)\,\mathrm{d}y}{x-y}.$$

The integral term of Eq. (1.1) becomes

$$\oint_{-1}^{1} \frac{\phi(\tau)}{x-\tau}\mathrm{d}\tau = 2\sum_{n=0}^{\infty} C_n Q_n(x). \tag{3.11}$$

The term $(1-x^2)^{1/2}$ is called the weight function of the integro-differential equation, and f_n are constants that can be determined. Also $P_n^m(x)$ is the associated Legendre Polynomial of the first kind, while $Q_n(x)$ is the Legendre Polynomial of the second kind which satisfies the famous relation (see [10, Eq. 7112-2, pp. 808])

$$\int_{-1}^{1} Q_n^k(x)P_m^k(x)\,\mathrm{d}x = (-1)^k \frac{1-(-1)^{n+m}(n+k)!}{(m-n)(m+n+1)(n-k)!}.$$

Using the associated orthogonal Legendre relation (see [10, Eq. 7112. pp. 808])

$$\int_{-1}^{1} P_n^k(x)P_m^k(x)\,\mathrm{d}x = \begin{cases} 0 & n \neq m, \\ \dfrac{2}{2n+1}\dfrac{(n+k)!}{(n-k)!} & n = m, \end{cases}$$

in (3.10) the expansion constants f_n take the form

$$f_n = \frac{-(1+2n)}{2n(n+1)} \int_{-1}^{1} (1-x^2)^{1/2} f(x) P_n^1(x)\,\mathrm{d}x. \tag{3.12}$$

The polynomial series (3.10) is uniformly convergent to $f(x)$ in $L_2[-1,1]$, if $f(x) \in L_2([-1,1])$. Using (3.9)–(3.11) in (1.1), we have

$$2\lambda\sqrt{(1-x^2)} \sum_{n=0}^{\infty} C_n Q_n(x) = \sum_{n=1}^{\infty} (f_n - \mu C_n) P_n^1(x). \tag{3.13}$$

The above expression contains a system of Linear algebra in the associated Legendre polynomial of the first and of the second kind. To obtain the compact form of the constants C_n, we use the following relations (see [7], [10, pp. 807]):

$$m(m+1)[P_{m+1}(x) - P_{m-1}(x)] = (2m+1)\sqrt{(1-x^2)}P_m^1(x)$$

and

$$\int_{-1}^{1} \sqrt{(1-x^2)} Q_n(x) P_m^1(x)\,\mathrm{d}x = \begin{cases} 0 & (n = m \pm 1) \\ \dfrac{-2m(m+1)[1+(-)^{n+m}]}{(m-n-1)(m-n+1)(m+n)(m+n+2)} & (n \neq m \pm 1) \end{cases}$$

formula (3.13) takes the form

$$\mu C_m + 2\lambda \sum_{n=0, n\neq m\pm 1}^{\infty} C_n \frac{-(2m+1)[1+(-1)^{n+m}]}{(m-n-1)(m-n+1)(m+n)(m+n+2)} = f_m$$

$(C_0 = p/2, m \geqslant 1)$.

The last relation represents an infinite system of linear equations; we can rewrite it to obtain

$$\mu X_{2i} + \lambda \sum_{n=1}^{\infty} X_{2n} b_{2i,2n} = G_{2i} - p b_{2i,0} \tag{3.14}$$

and

$$\mu X_{2i-1} + \lambda \sum_{n=1}^{\infty} X_{2n-1} b_{2i-1,2n-1} = G_{2i-1}, \tag{3.15}$$

where

$$X_i = C_i(2i+1)^{-1}, \quad G_i = f_i(2i+1)^{-1} \quad (i \geqslant 1),$$

$$b_{i,n} = \frac{-2(2n+1)[1+(-1)^{i+n}]}{(i-n-1)(i-n+1)(i+n)(i+n+2)}$$

and

$$b_{i,i-1} = b_{i,i+1} = 0.$$

For $c \geqslant 1$, it is not difficult to obtain

$$\sum_{n=1}^{\infty} |b_{in}| < S,$$

$$S = \frac{3}{2} + 4\sum_{n=3}^{\infty} \frac{2n+1}{(n-2)n(n+1)(n+3)} = \frac{3}{2} + 4(\beta_1 + \beta_2),$$

where

$$\beta_1 = \sum_{j=0}^{\infty} \frac{1}{(j+1)(j+4)(j+6)} = -\frac{1}{15}\psi(1) + \frac{1}{6}\psi(4) - \frac{1}{10}\psi(6),$$

$$\beta_2 = \sum_{j=0}^{\infty} \frac{1}{(m+1)(m+3)(m+6)} = -\frac{1}{10}\psi(1) + \frac{1}{6}\psi(3) - \frac{1}{15}\psi(6),$$

where $\psi(n)$ is called Psi function (see [6,7]). Taking the transformations $y = 2z - 1$; Eq. (1.1) becomes

$$\mu \frac{d\Theta}{dt} - \lambda \oint_0^1 \frac{\Theta(z)\,dz}{t-z} = f(t), \quad t \in (0,1). \tag{3.16}$$

This equation has appeared both in combined infrared gaseous radiation and molecular conduction, where the physical parameter λ in (3.16), known as the radiation–conduction number for the large path length limit, represents the single parameter of the dimensionless system. Eq. (3.16) is solved in [9] when $f(t) = -t + 1/2$ under the condition $\Theta(0) = \Theta(1) = 0$ where Θ represents the unknown temperature. If $\mu = 0$ in (3.16), then the derivative term vanishes and the classical airfoil equation is recovered. Also if $\mu = 0$ in Eq. (2.6) we have the integral equation of anti-plane deformation for the displacement problems, for an infinite rigid strip with width $2a$ when putting on an elastic layer of thickness h (see [1]). If Eq. (2.7) for $\alpha > 1$ is used in (2.6) the integral equation of fracture mechanics often arises (see [14]).

4. Discussion and numerical examples

In this section we give some numerical examples to show how our procedure deals with equation such as Eq. (1.1) under condition (1.2). We truncate expansions (3.8) and (3.10) to N-terms, namely

$$\phi(x) \simeq \phi_N(x) = \sum_{n=0}^{N} C_n P_n(x),$$

$$f(x) \simeq f_N(x) = -\sum_{n=0}^{\infty} f_n (1-x^2)^{-1/2} P_n^1(x).$$

It is easy to see that $\phi_N(x)$ satisfies the equation

$$\mu\frac{\mathrm{d}\phi_N}{\mathrm{d}x} - \lambda \oint_{-1}^{1} \frac{\phi_N(y)\,\mathrm{d}y}{x-y} = f_N(x), \quad |x| \leqslant 1, \quad C_0 = p/2.$$

The L_2-error after N terms is

$$\|\phi - \phi_N\|_2 = \sum_{n=N+1}^{\infty} |C_N^2|.$$

The orthogonality of the bases $P_n(x)$ implies

$$C_N = \int_{-1}^{1} \phi(x) P_n(x)\,\mathrm{d}x$$

therefore,

$$\begin{aligned} |C_n| &\leqslant \|\phi\|_2 \|P_n\|_2 \\ &\leqslant \frac{2\|\phi\|_2}{2n+1} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This ensures that $\|\phi - \phi_N\|_2 \to 0$ as $N \to 0$.

Throughout the following three examples we assume $p = 1$ and $\lambda = 0.1$.

Example 1. Let $f(x) = x^2$ then Eq. (3.8) gives

$$\phi(x) = 0.5 + \sum_{n \text{ even}} C_n P_n(x) + \sum_{n \text{ odd}} C_n P_n(x).$$

The coefficients C_n are listed in Table 1 for $\mu = 0.01$ and in Table 2 for $\mu = 0.9$.

Example 2. Let $f(x) = x$ then Eq. (3.8) gives

$$\phi(x) = 0.5 + \sum_{n \text{ even}} C_n P_n(x).$$

The coefficients C_n for n even only exist but the odd ones are zeros and are listed in Table 1 for $\mu = 0.01$ and in Table 2 for $\mu = 0.9$.

Example 3. Let $f(x) = 0$ then Eq. (3.8) gives

$$\phi(x) = 0.5 + \sum_{n \text{ even}} C_n P_n(x).$$

The coefficients C_n for n odd are all zeros, while the others are also listed in Table 1 for $\mu = 0.01$ and in Table 2 for $\mu = 0.9$.

Table 1
C-coefficients of Eq. (3.8) for $\mu = 0.01$

C_n	$f(x)=0$	$f(x)=x$	$f(x)=x^2$
C_1	0.0	−0.4144435810	0.0
C_2	1.1780383910	1.178038391	1.081396844
C_3	0.0	−0.2180344723	0.0
C_4	1.078603786	1.0786037850	1.075225186
C_5	0.0	−0.202694007	0.0
C_6	0.9534920469	0.9534920473	0.9463089248
C_7	0.0	−0.1800439305	0.0
C_8	0.8258473710	0.8258473707	0.8176992149
C_9	0.0	−0.1559101971	0.0
C_{10}	0.7043685835	0.7043685828	0.694014139
C_{11}	0.0	−0.1324681567	0.0
C_{12}	0.5925588353	0.5925588350	0.582720786
C_{13}	0.0	−0.114700108	0.0
C_{14}	0.4908995049	0.4908995046	0.485104285
C_{15}	0.0	−0.08982746067	0.0
C_{16}	0.3974779320	−0.3974779318	0.3920878180
C_{17}	0.0	−0.06956586798	0.0
C_{18}	0.3071175318	0.3071175317	0.3028219847
C_{19}	0.0	−0.04650761977	0.0
C_{20}	0.2052591874	0.205251873	0.2023194624

Table 2
C-coefficients of Eq. (3.8) for $\mu = 0.9$

C_n	$f(x)=0$	$f(x)=x$	$f(x)=x^2$
C_1	0.0	−0.1787846010	0.0
C_2	0.3029501545	0.3209501543	0.2902650023
C_3	0.0	−0.005299917974	0.0
C_4	0.08141206582	0.08141206577	0.08849553584
C_5	0.0	−0.002959259702	0.0
C_6	0.02731884568	0.02731884568	0.02865089844
C_7	0.0	−0.00144233513	0.0
C_8	0.01182792997	0.01182792997	0.01219580830
C_9	0.0	−0.0007749453207	0.0
C_{10}	0.006106290323	0.006106290324	0.006235437271
C_{11}	0.0	−0.004581513471	0.0
C_{12}	0.003550024663	0.003550024661	0.003602518027
C_{13}	0.0	−0.0002918927134	0.0
C_{14}	0.002243305937	0.002243305937	0.002266550625
C_{15}	0.0	−0.0001969234851	0.0
C_{16}	0.00150686507	0.001506786508	0.0015174886975
C_{17}	0.0	−0.0001387858164	0.0
C_{18}	0.001059329243	0.001059329242	0.001064200708
C_{19}	0.0	−0.000100360217	0.0
C_{20}	0.0007652569118	0.0007652569117	0.0007672658421

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